

Задача 37. Экстремумы функции

$$F(x, y, z) = 0$$

$$F'_x dx + F'_y dy + F'_z dz = 0 \Rightarrow F'_x = 0, F'_y = 0$$

$$dz = - \frac{F'_x dx + F'_y dy}{F'_z} = 0 \Rightarrow \text{используем}$$

$$\begin{aligned} d^2 z &= - \frac{(F''_{xx} dx^2 + 2F''_{xy} dx dy + F''_{yy} dy^2) \cdot F'_z + (F''_{xz} dx + F''_{yz} dy) \left(- \frac{F'_x dx + F'_y dy}{F'_z} \right)}{(F'_z)^2} - \\ &= - \frac{(F''_{xx} dx^2 + 2F''_{xy} dx dy + F''_{yy} dy^2) F'_z + (F''_{xz} dx + F''_{yz} dy) \left(- \frac{F'_x dx + F'_y dy}{F'_z} \right)}{(F'_z)^2} = - \frac{F''_{xx} dx^2 + 2F''_{xy} dx dy + F''_{yy} dy^2}{(F'_z)^2} \quad (*) \end{aligned}$$

N 109 d) Доказать на примере $z(x, y)$

$$F(x, y, z) = x^2 + y^2 + z^2 - xz - yz + 2(x + y + z - 1)$$

Решение. $\begin{cases} F'_x = 2x - z + 2 = 0 \\ F'_y = 2y - z + 2 = 0 \\ F'_z = 2z - x - y + 2 = 0 \end{cases} \Rightarrow \begin{cases} z = 2x + 2 \\ z = 2y + 2 \\ y = x \end{cases}$

$$\begin{aligned} \Rightarrow x^2 + x^2 + (2x+2)^2 - x(2x+2) - x(2x+2) + 2(x+x+2x+2-1) &= 0 \\ 2x^2 + 4(x^2 + 2x + 1) - 4x^2 - 4x + 8x + 2 &= 0 \\ 2x^2 + 12x + 6 &= 0 \quad x^2 + 6x + 3 = 0 \\ D = 36 - 12 = 24, \quad x = y = -3 \pm \sqrt{6} \end{aligned}$$

$$M_1(-3 + \sqrt{6}, -3 + \sqrt{6}), \quad z_1 = -4 + 2\sqrt{6}$$

$$M_2(-3 - \sqrt{6}, -3 - \sqrt{6}), \quad z_2 = -4 - 2\sqrt{6}$$

$$dF = 2x dx + 2y dy + 2z dz - z dx - x dz - z dy - y dz + 2(dx + dy + dz) = 0$$

$$d^2 F = 2dx^2 + 2dy^2 + 2dz^2 + 2z dx - 2z dy - (x+y) dz^2 + 2d^2 z = 0$$

$$\Rightarrow d^2 z = \frac{2dx^2 + 2dy^2 + 2dz^2 - 2dz(dx + dy)}{2z + 2 - (x + y)}$$

$$F''_{xx} = 2, \quad F''_{xy} = 0, \quad F''_{yy} = 2, \quad F'_z = 2z - x - y + 2$$

$$\Rightarrow \text{по формуле (*) } d^2 z = - \frac{2dx^2 + 2dy^2}{2z - x - y + 2} \quad d^2 z(M_1) =$$

$$= - \frac{2dx^2 + 2dy^2}{-8 \pm 4\sqrt{6} + 6 \mp 2\sqrt{6} + 2} = - \frac{2dx^2 + 2dy^2}{\pm 2\sqrt{6}} = \mp \frac{dx^2 + dy^2}{\sqrt{6}} \Rightarrow \begin{matrix} M_1 - \text{max} \\ M_2 - \text{min} \end{matrix}$$

Условия экстремумов

Пусть $f, g_j, j \in \overline{1, r}$ — конт. функ. в окр. $x_0 \in \mathbb{R}^m$, $f, g_j : \mathbb{R}^m \rightarrow \mathbb{R}$,
 $g_j(x_0) = 0$ — конт. ф-ция в окр. x_0 .

Пусть $f, g_j (j \in \overline{1, r}) : \mathbb{R}^m \rightarrow \mathbb{R}$,

$$g_j(x_0) = 0.$$

Рассмотрим функцию $F = f(x) + \sum_{j=1}^r \lambda_j g_j(x)$

Если f и g_j — конт. функ. в окр. $x_0 \in \mathbb{R}^m$,

$$g_j'(x_0) = 0 \quad \forall j \in \overline{1, r}, \quad \text{при конт. ф-ции}$$

тогда необходимо условие экстремума

$$\text{в окр. } x_0 : \begin{cases} \frac{\partial F}{\partial x_i} = 0 & i \in \overline{1, m} \\ \frac{\partial F}{\partial \lambda_j} = 0 & j \in \overline{1, r} \end{cases}$$

[N 110 8] Докажем на условиях экстремума

$$f(x, y) = x^2 + 2y^2 - xy, \quad g_1(x, y) = x^2 + y^2 - 4 = 0$$

Решение. $F = x^2 + 2y^2 - xy + \lambda(x^2 + y^2 - 4)$

$$\begin{cases} F'_x = 2x - y + 2\lambda x = 0 \\ F'_y = 4y - x + 2\lambda y = 0 \\ F'_\lambda = x^2 + y^2 - 4 = 0 \end{cases} \Rightarrow \begin{cases} (2+2\lambda)x - y = 0 \\ -x + (4+2\lambda)y = 0 \end{cases}$$

Контр. разб., когда $\Delta \neq 0$

$$\Rightarrow (2+2\lambda)(4+2\lambda) - 1 = 0$$

$$4\lambda^2 + 12\lambda + 8 - 1 = 0$$

$$\Rightarrow 4\lambda^2 + 12\lambda + 7 = 0$$

$$\Delta = 144 - 112 = 32$$

$$\lambda_{1,2} = \frac{-12 \pm 2\sqrt{32}}{8} = \frac{-3 \pm \sqrt{2}}{2}$$

$$\Rightarrow x = \frac{(4 + \frac{-3 \pm \sqrt{2}}{2})y}{1} = \frac{5 \pm \sqrt{2}}{2}y$$

$$\left(\frac{5 \pm \sqrt{2}}{2}y\right)^2 + y^2 - 4 = 0 \Rightarrow \lambda_{1,2} = 12 \pm 4\sqrt{3} \quad 3 \pm \sqrt{3}$$

$$\lambda_{1,2} = \frac{-12 \pm \sqrt{32}}{8} = \frac{-12 \pm 4\sqrt{2}}{8} = \frac{-3 \pm \sqrt{2}}{2}$$

$$x = \left(4 + \frac{-3 \pm \sqrt{2}}{2} \cdot 2\right)y = (4 + (-3 \pm \sqrt{2}))y = (1 \pm \sqrt{2})y$$

$$(1 \pm \sqrt{2})^2 y^2 + y^2 = 4 \Rightarrow y = \pm \frac{2}{\sqrt{(1 \pm \sqrt{2})^2 + 1}} = \pm \frac{2}{\sqrt{4 \pm 2\sqrt{2}}}$$

$$\Rightarrow x = \pm \frac{2(1 \pm \sqrt{2})}{\sqrt{4 \pm 2\sqrt{2}}}$$

~~$\Rightarrow M_1(2)$~~

$$\lambda_1 = \frac{-3 + \sqrt{2}}{2}$$

$$M_1 \left(\frac{2 + 2\sqrt{2}}{\sqrt{4 + 2\sqrt{2}}}, \frac{2}{\sqrt{2 + 2\sqrt{2}}} \right)$$

$$M_2 \left(-\frac{2 + 2\sqrt{2}}{\sqrt{4 + 2\sqrt{2}}}, -\frac{2}{\sqrt{2 + 2\sqrt{2}}} \right)$$

$$dF(M_{1,2}) = 2x dx + 4y dy - (y dx + x dy) + (-3 + \sqrt{2})(x dx + y dy)$$

$$= (-1 + \sqrt{2})x dx - y dx + (1 + \sqrt{2})y dy - x dy$$

$$d^2(M_{1,2}) = (-1 + \sqrt{2})dx^2 + (1 + \sqrt{2})dy^2 - 2dx dy$$

$$\Delta_1 > 0, \Delta_2 = (-1 + \sqrt{2})(1 + \sqrt{2}) - 1 = 0$$

$$\equiv (\sqrt{-1 + \sqrt{2}} dx + \sqrt{1 + \sqrt{2}} dy)^2 \geq 0 \quad - M_{1,2} - \text{min}$$

$$\lambda_2 = \frac{-3 - \sqrt{2}}{2}$$

$$M_3 \left(\frac{2 - 2\sqrt{2}}{\sqrt{4 - 2\sqrt{2}}}, \frac{2}{\sqrt{2 - 2\sqrt{2}}} \right)$$

$$M_4 \left(-\frac{2 - 2\sqrt{2}}{\sqrt{4 - 2\sqrt{2}}}, -\frac{2}{\sqrt{2 - 2\sqrt{2}}} \right)$$

$$d^2 F(M_{3,4}) = (-1 - \sqrt{2})dx^2 + (1 - \sqrt{2})dy^2 - 2dx dy =$$

$$= -((\sqrt{2} + 1)dx^2 + 2dx dy + (\sqrt{2} - 1)dy^2) = -(\sqrt{2} + 1 dx + \sqrt{2} - 1 dy)^2$$

$$\leq 0 \quad - M_{3,4} - \text{lok. MAX.}$$

[N 110 4-3]

$$f(x, y, z) = xyz$$

$$x^2 + y^2 + z^2 = 1$$

$$x + y + z = 0$$

Решение. Ф-я Лагранжа $F = xyz + \lambda(x^2 + y^2 + z^2 - 1) + \mu(x + y + z)$

$$F'_x = yz + 2\lambda x + \mu = 0$$

$$F'_y = xz + 2\lambda y + \mu = 0$$

$$F'_z = xy + 2\lambda z + \mu = 0$$

$$x^2 + y^2 + z^2 = 1$$

$$x + y + z = 0$$

$$(x + y + z)^2 - (x^2 + y^2 + z^2) = 2(yz + xz + xy) = -1 \Rightarrow 3\mu = \frac{1}{2} \Rightarrow \mu = \frac{1}{6}$$

$$(x - y)z - 2\lambda(x - y) = 0$$

$$(x - y)(z - 2\lambda) = 0$$

$$(x - z)(y - 2\lambda) = 0$$

$$(y - z)(x - 2\lambda) = 0$$

$$\Rightarrow x = y = 2\lambda$$

$$z = -4\lambda$$

$$x^2 + y^2 + z^2 = 1$$

$$\text{ad. } \begin{cases} y = z = 2\lambda \\ x = -4\lambda \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

$$\text{ad. } \begin{cases} x = z = 2\lambda \\ y = -4\lambda \\ x^2 + y^2 + z^2 = 1 \end{cases} \Rightarrow$$

$$4\lambda^2 + 4\lambda^2 + 16\lambda^2 = 1 \Rightarrow 24\lambda^2 = 1, \quad \lambda = \pm \frac{1}{2\sqrt{6}}$$

$$\Rightarrow M_{1,2} = \left(\pm \frac{1}{\sqrt{6}}, \pm \frac{1}{\sqrt{6}}, \mp \frac{2}{\sqrt{6}} \right)$$

$$M_{3,4} = \left(\pm \frac{1}{\sqrt{6}}, \mp \frac{2}{\sqrt{6}}, \pm \frac{1}{\sqrt{6}} \right)$$

$$M_{5,6} = \left(\mp \frac{2}{\sqrt{6}}, \pm \frac{1}{\sqrt{6}}, \pm \frac{1}{\sqrt{6}} \right)$$

$$dF = yz dx + xz dy + xy dz + 2\lambda(x dx + y dy + z dz) + \mu(dx + dy + dz)$$

$$\cancel{d^2F = 2z dy dz + 2y dz dx + 2x dx dy + 2\lambda(dx^2 + dy^2 + dz^2) + \mu(dx + dy + dz)}$$

$$\begin{cases} d^2F = 2z dx dy + 2y dx dz + 2x dy dz + 2\lambda(dx^2 + dy^2 + dz^2) \\ x dx + y dy + z dz = 0 \\ dx + dy + dz = 0 \end{cases}$$

$$M_{1,2} : d^2F(M_{1,2}) = \left. \begin{matrix} x = y = 2\lambda \\ z = -4\lambda \end{matrix} \right| = 2\lambda(dx^2 + dy^2 + dz^2) + \cancel{2\lambda dx + 2\lambda dy}$$

$$- 8\lambda dx dy + 4\lambda dz dx + 4\lambda dy dz =$$

$$= 2\lambda(dx^2 + dy^2 + dz^2 - 4 dx dy + 2 dz dx + 2 dy dz) =$$

$$= \left. \frac{dz = -dx - dy}{dz = -dx - dy} \right| = 2\lambda(dx^2 + dy^2 + dz^2 - 4 dx dy + 2(dx - dy)^2)$$

$$= 2\lambda(dx^2 + dy^2 + dz^2 + 2(dx - dy)^2)$$

$$\geq 0 \quad \text{b. r. } M_1 - \min$$

$$\leq 0 \quad \text{b. r. } M_2 - \max$$

$$\text{Аналогично } \text{b. r. } M_3 - \min$$

$$M_4 - \max$$

$$M_5 - \min$$

$$M_6 - \max$$

N 110 K)

$$f(x, y, z) = xy + yz, \quad x^2 + y^2 - 2 = 0, \quad y + z - 2 = 0$$

$$x > 0, y > 0, z > 0,$$

Розв'язок. $F = xy + yz + \lambda(x^2 + y^2 - 2) + \mu(y + z - 2)$

$$\begin{cases} F'_x = y + 2\lambda x = 0 \\ F'_y = x + z + 2\lambda y + \mu = 0 \\ F'_z = y + \mu = 0 \\ x^2 + y^2 = 2 \\ y + z = 2 \end{cases}$$

$$\Rightarrow x = y = z = 1, \quad \lambda = -\frac{1}{2}, \quad \mu = -1$$

$$dF = ydx + xdy + zdy + ydz + 2\lambda(xdx + ydy) + \mu(dy + dz)$$

$$d^2F = 2\lambda(dx^2 + dy^2) + 2dxdy + 2dydz = \left/ \lambda = -\frac{1}{2} \right/ =$$

$$= -dx^2 - dy^2 + 2dxdy + 2dydz \quad \text{⊖}$$

3 півч. зв'язок $\Rightarrow dy = -dz$

$$\text{⊖} \quad -dx^2 - dy^2 + 2dxdy - 2dy^2 = -(dx - dy)^2 - 2dy^2 \leq 0$$

$$\Rightarrow \text{в т. } (1, 1, 1) \text{ функція має максимум } f = 2.$$

N 111 8)

$$f(x, y) = x^2 + y^2 - 12x + 16y, \quad D = \{(x, y) \mid x^2 + y^2 \leq 25\}$$

Знайти верхню та нижню грани (набд. та найменше знач.) на множині D.

Розв'язок. f - непер. у замкн. обмеж. множині D

$\Rightarrow$ за теор. Вейерштрасса f досягає на D точних верхньої та нижньої грани (набд. та найм. значення)

$\Rightarrow$ Набд. / найменше знач. досягаються або у локальному екстр. на $\overline{x^2 + y^2}$ внутрішності D: $x^2 + y^2 < 25$

або в точках умовного екстремуму при $x^2 + y^2 = 25$.

$$\begin{cases} F'_x = 2x - 12 = 0 \\ F'_y = 2y + 16 = 0 \end{cases} \Rightarrow \begin{matrix} x = 6 \\ y = -8 \end{matrix} \quad \text{Але } 6^2 + (-8)^2 > 25$$

$\Rightarrow \sup f$ та $\inf f$ досягаються при $x^2 + y^2 = 25$

Функція Лагранжа $F = x^2 + y^2 - 12x + 16y + \lambda(x^2 + y^2 - 25)$

$$\begin{cases} F'_x = 2x - 12 + 2\lambda x = 0 \\ F'_y = 2y + 16 + 2\lambda y = 0 \\ x^2 + y^2 = 25 \end{cases}$$

$$\Rightarrow \lambda = -1, \quad M_1(3, -4), \quad M_2(-3, 4)$$

$$z(M_1) = -75, \quad z(M_2) = 125$$

$$\begin{matrix} \parallel & \parallel \\ \min & \max \end{matrix}$$

N111 2) $f(x, y, z) = x + y + z$, $D: x^2 + y^2 \leq z \leq 1$

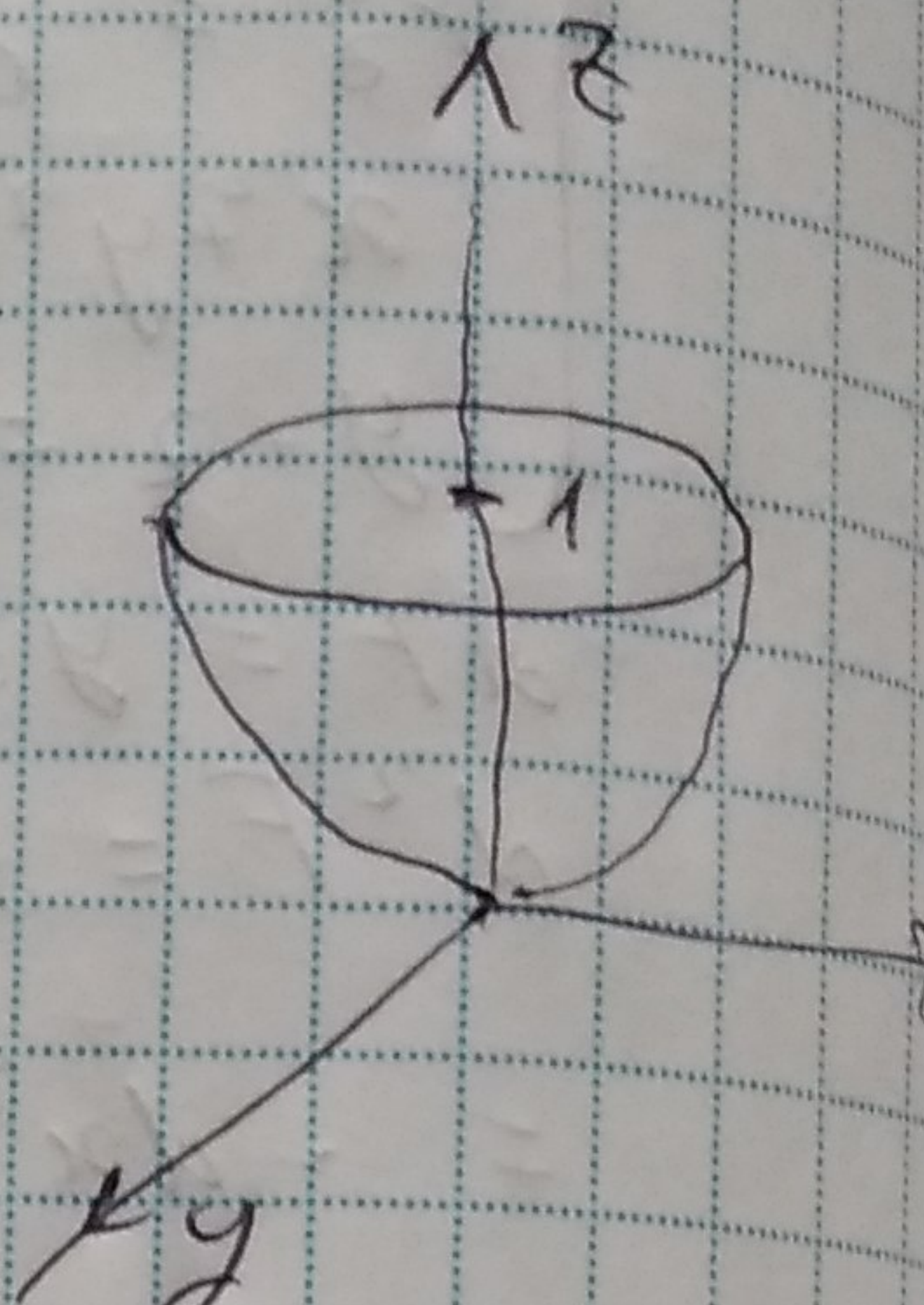
Решение. Поскольку, f не имеет точек безусловной экстремума (внутри области) $\Rightarrow$ достигн. экстр. на границах области $x^2 + y^2 = z$

або при $z = 1$:

1) $\begin{cases} f = x + y + z \\ z = 1 \\ x^2 + y^2 \leq 1 \end{cases}$ (Дост. экстр. на окружности)

$$\begin{aligned} F &= x + y + 1 + \lambda(x^2 + y^2 - 1) \\ \begin{cases} F'_x = 1 + 2\lambda x = 0 \\ F'_y = 1 + 2\lambda y = 0 \\ x^2 + y^2 = 1 \end{cases} &\Rightarrow \begin{cases} x = -\frac{1}{2\lambda} \\ y = -\frac{1}{2\lambda} \end{cases} \\ \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} &= 1 \Rightarrow \lambda = \pm \frac{1}{\sqrt{2}} \Rightarrow x = y = \mp \frac{1}{\sqrt{2}} \end{aligned}$$

$\Rightarrow M_1(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1)$, $M_2(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1)$ — возможные точки экстр.



2) $\begin{cases} f = x + y + z \\ x^2 + y^2 = z < 1 \end{cases}$

$F = x + y + (x^2 + y^2)$

$$\begin{cases} f'_x = 1 + 2x = 0 \\ f'_y = 1 + 2y = 0 \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{2} \\ y = -\frac{1}{2} \\ z = \frac{1}{2} \end{cases}$$

$\Rightarrow M_3(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ — возможная экстр.

$f(M_1) = -\sqrt{2} + 1$, $f(M_2) = \sqrt{2} + 1$, $f(M_3) = -\frac{1}{2}$

$\Rightarrow$ в M_2 — max sup

в M_3 — min. inf

D/3. N109 a), N110 a) g) e) m), N111 a) g) n)