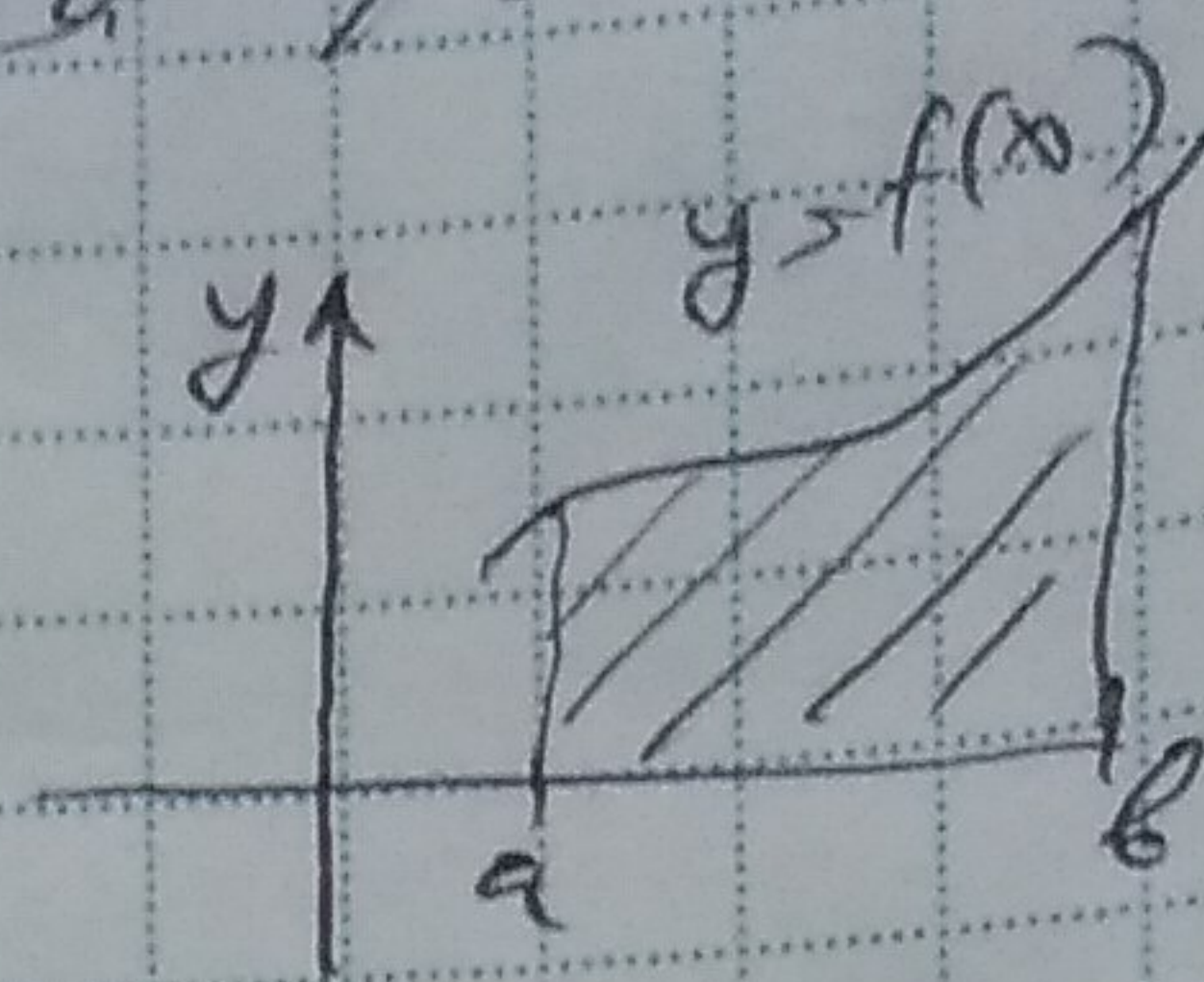


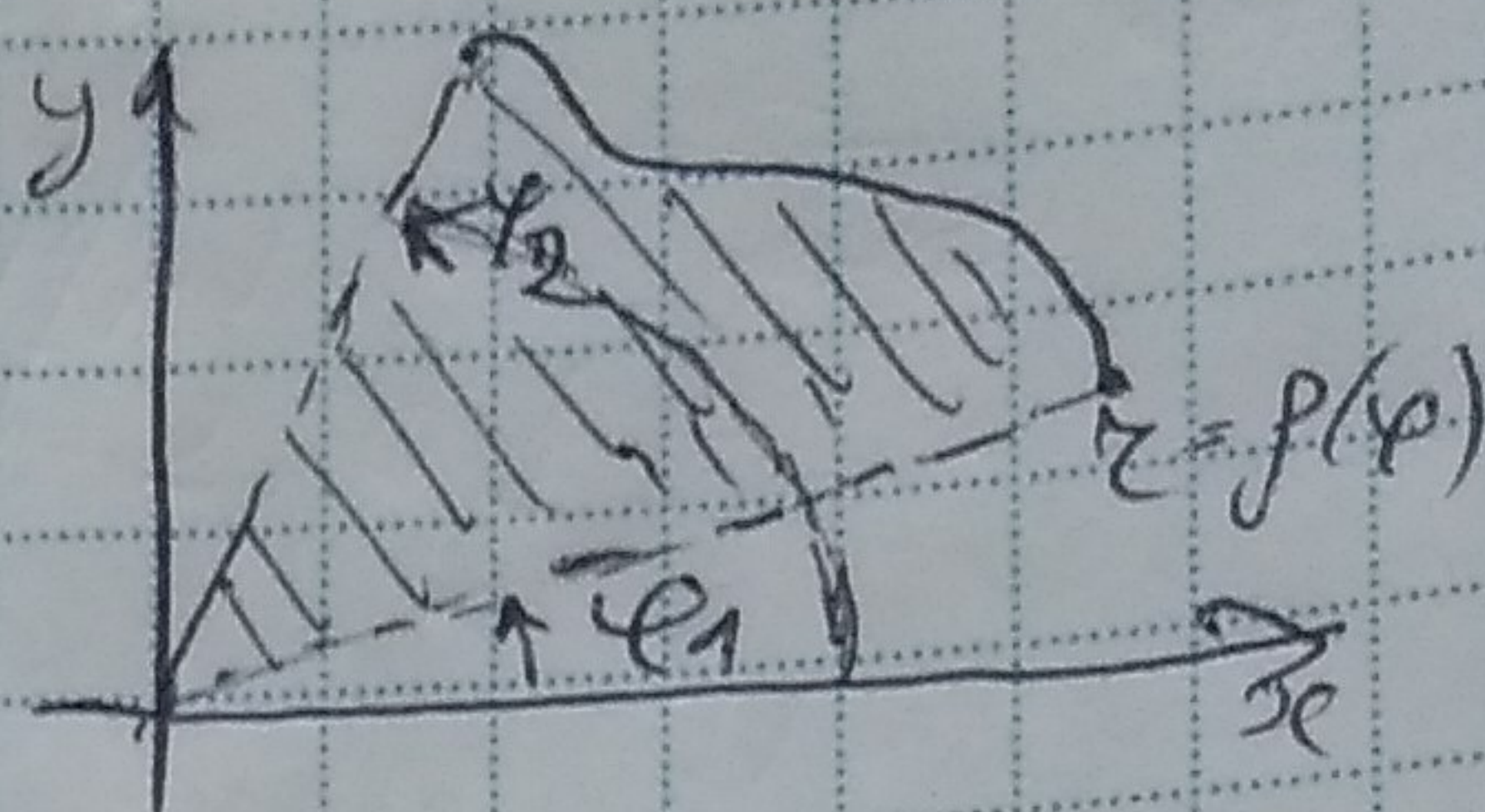
Заняття 29

Застосування інтеграла Ріманна

1. Площа криволінійної трапеції
 $D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, 0 \leq y \leq f(x)\}$
 $S = \int_a^b f(x) dx$



2. Площа криволінійного сектора
 $D = \{(r, \varphi) \in \mathbb{R}^2 \mid 0 \leq r \leq \rho(\varphi), \varphi_1 \leq \varphi \leq \varphi_2\}$
 $S = \frac{1}{2} \int_{\varphi_1}^{\varphi_2} \rho^2(\varphi) d\varphi$



3. Об'єм тіла обертання — тіла, що утворюється обертанням криволінійної трапеції:
 $D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, 0 \leq y \leq f(x)\}$, де
 f обертається навколо осі Ox

$$V = \pi \int_a^b f^2(x) dx$$

4. Об'єм тіла за відомих поперезках перерізом $S(x)$ при кожному фіксованому $x \in [a, b]$:

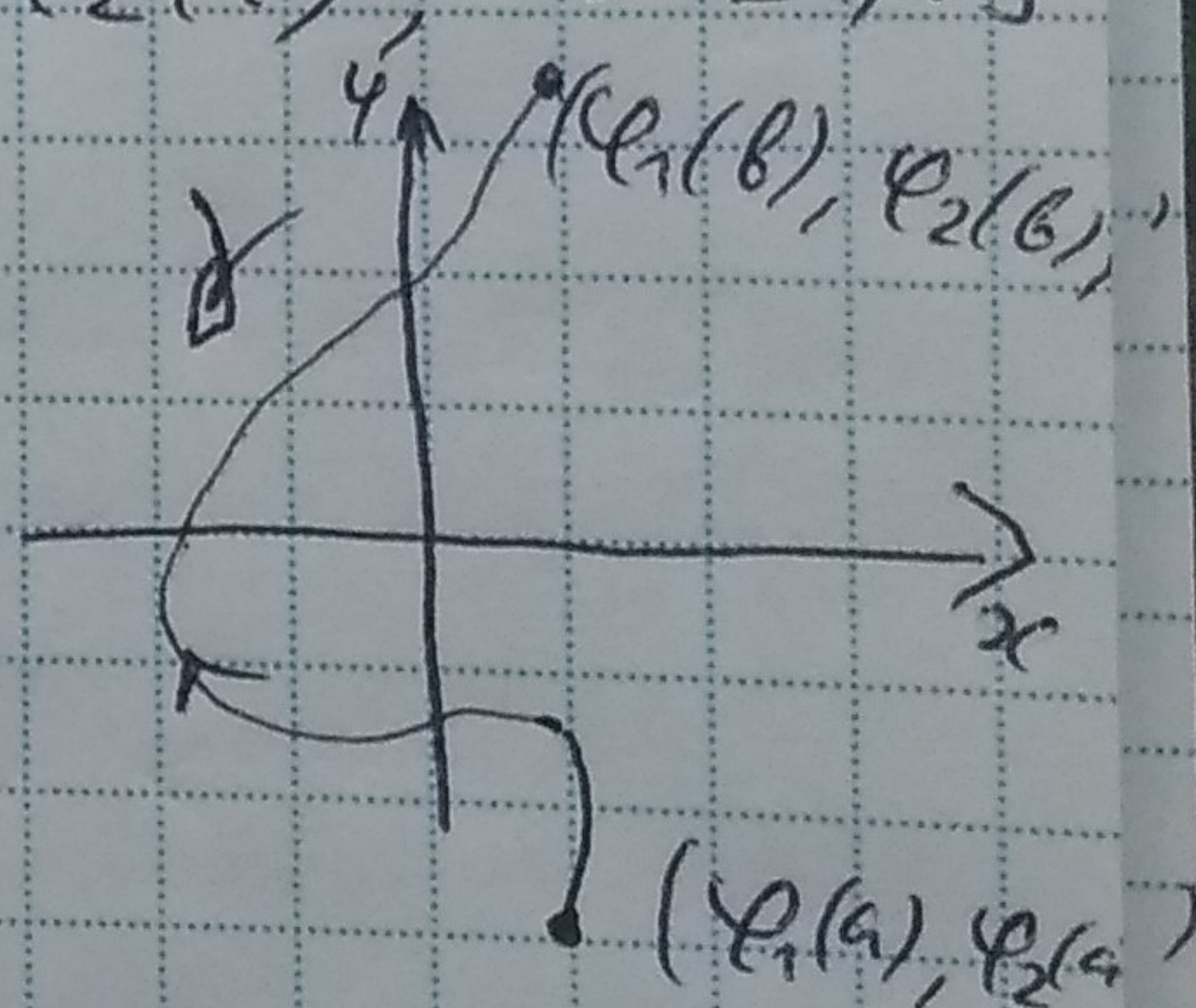
$$V = \int_a^b S(x) dx$$

5. Довжина кривої γ а) $x = \varphi_1(t)$, $y = \varphi_2(t)$, $t \in [a, b]$

$$L = \int_a^b \sqrt{(\varphi_1'(t))^2 + (\varphi_2'(t))^2} dt$$

б) якщо γ : $y = f(x)$, $x \in [a, b]$, то

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$



в) якщо γ : $\rho = \rho(\varphi)$, $\varphi \in [\varphi_1, \varphi_2]$ (в полярних координатах)

$$L = \int_{\varphi_1}^{\varphi_2} \sqrt{\rho^2(\varphi) + (\rho'(\varphi))^2} d\varphi$$

№ 159 а) Знайти площу фігури, обмеженою

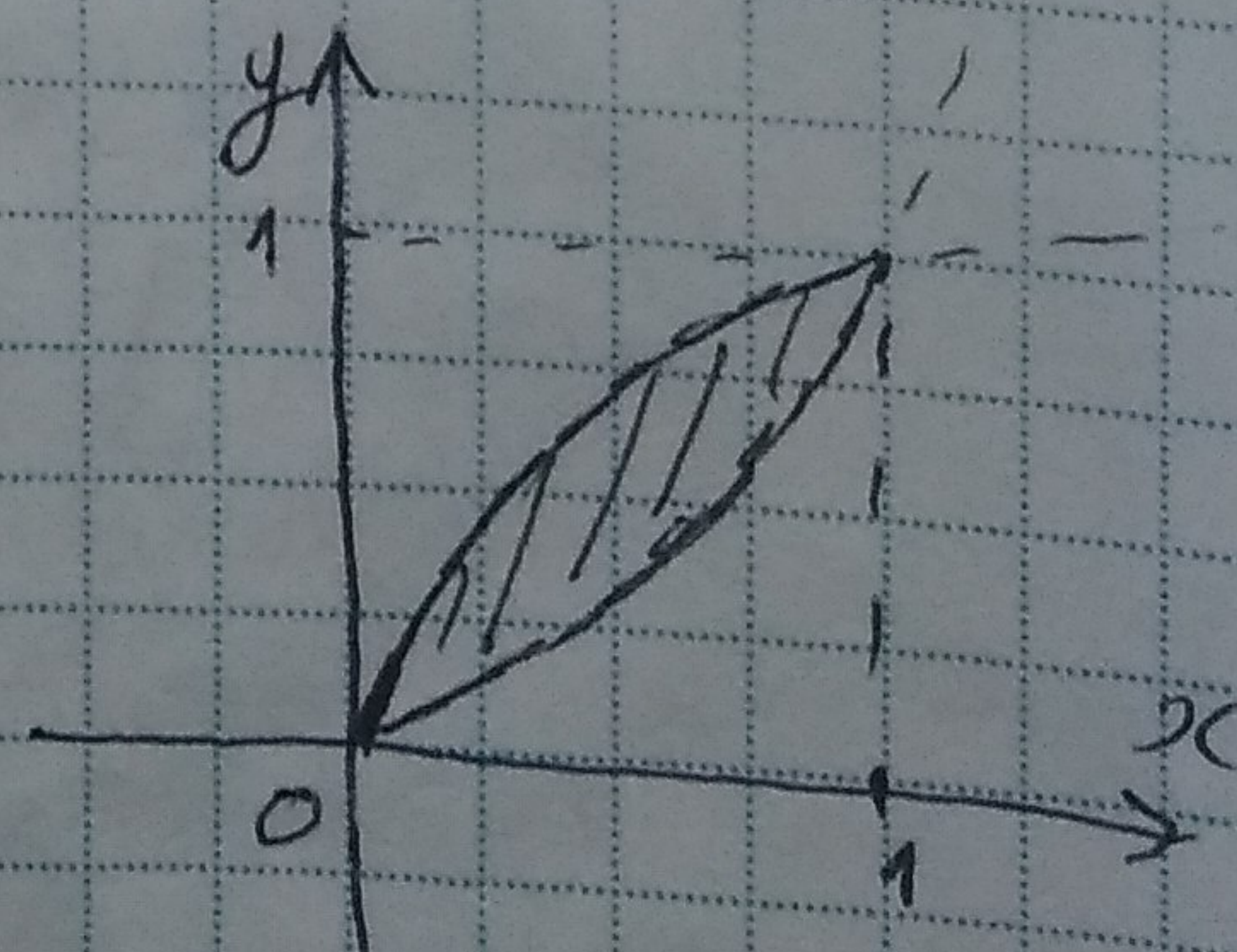
кривими $x = y^2$, $x^2 = y$

Розв'язок. $y = \pm \sqrt{x}$, $y = x^2$
 $f_1(x) = \sqrt{x}$, $f_2(x) = x^2$

Точка перетину кривих: $(0, 0)$, $(1, 1)$

$$\text{Площа } S = \int_0^1 (f_1(x) - f_2(x)) dx =$$

$$= \int_0^1 (\sqrt{x} - x^2) dx = \left. \frac{2x^{3/2}}{3} - \frac{x^3}{3} \right|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$



N 159 c)

Знайти площадь
 $y^2 = 2px, \quad 27py^2 = 8(x-p)^3$

Решение. Знаясьмо точки пересечения кривых

$$y^2 = 2px = \frac{8(x-p)^3}{27p}$$

$$54p^2x = (2(x-p))^3$$

Подбором: $x = cp \Rightarrow$
 $54c = 8(c-1)^3$
 $27c = 4(c-1)^3 \Rightarrow c = 4$

$\Rightarrow x = 4p \Rightarrow y^2 = 8p^2 \Rightarrow y = \pm \sqrt{8}p = \pm 2\sqrt{2}p$
 $f_1(x) = \sqrt{2px}, \quad f_2(x) = \frac{\sqrt{8}(x-p)^{3/2}}{\sqrt{27p}}$
 Область симметрична относительно осей Ox и Oy .

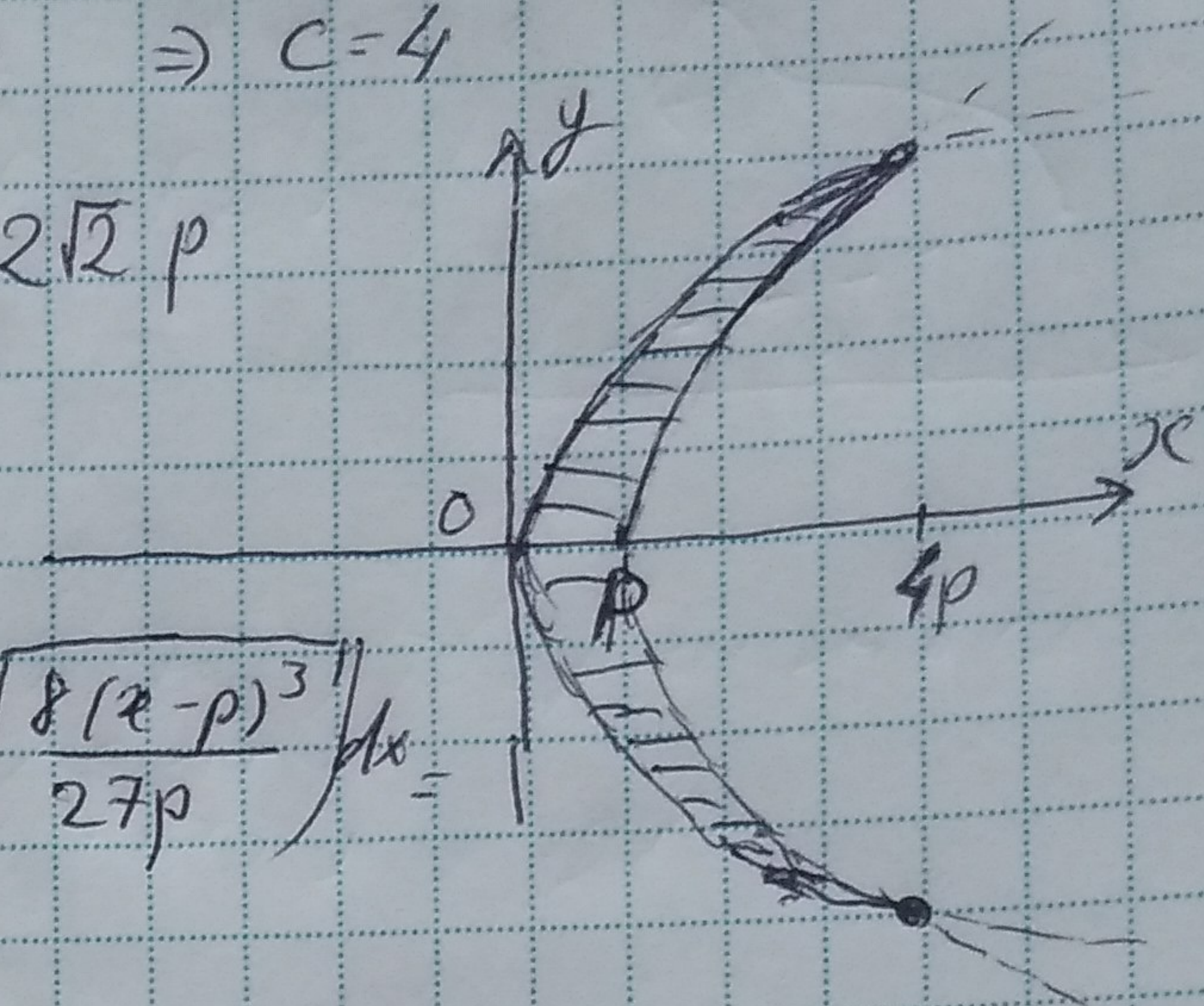
Вычисляем Ox , тогда $4p$

$$S = 2 \int_0^{4p} (f_1(x) - f_2(x)) dx = 2 \int_0^{4p} \left(\sqrt{2px} - \sqrt{\frac{8(x-p)^3}{27p}} \right) dx =$$

$$= 2 \int_0^{4p} \left(\sqrt{2p} \cdot x^{1/2} - \sqrt{\frac{8}{27p}} \cdot (x-p)^{3/2} \right) dx =$$

$$= 2 \left(\sqrt{2p} \cdot \frac{2}{3} (4p)^{3/2} - \sqrt{\frac{8}{27p}} \left(\frac{2(3p)^{5/2}}{5} + 2 \cdot \frac{p^{5/2}}{5} \right) \right) =$$

$$= \frac{32\sqrt{2}}{3} p^2 - \frac{4\sqrt{3}\sqrt{2}}{5} p^2 - \frac{4\sqrt{2}}{5\sqrt{27}} p^2$$



N 159 r)

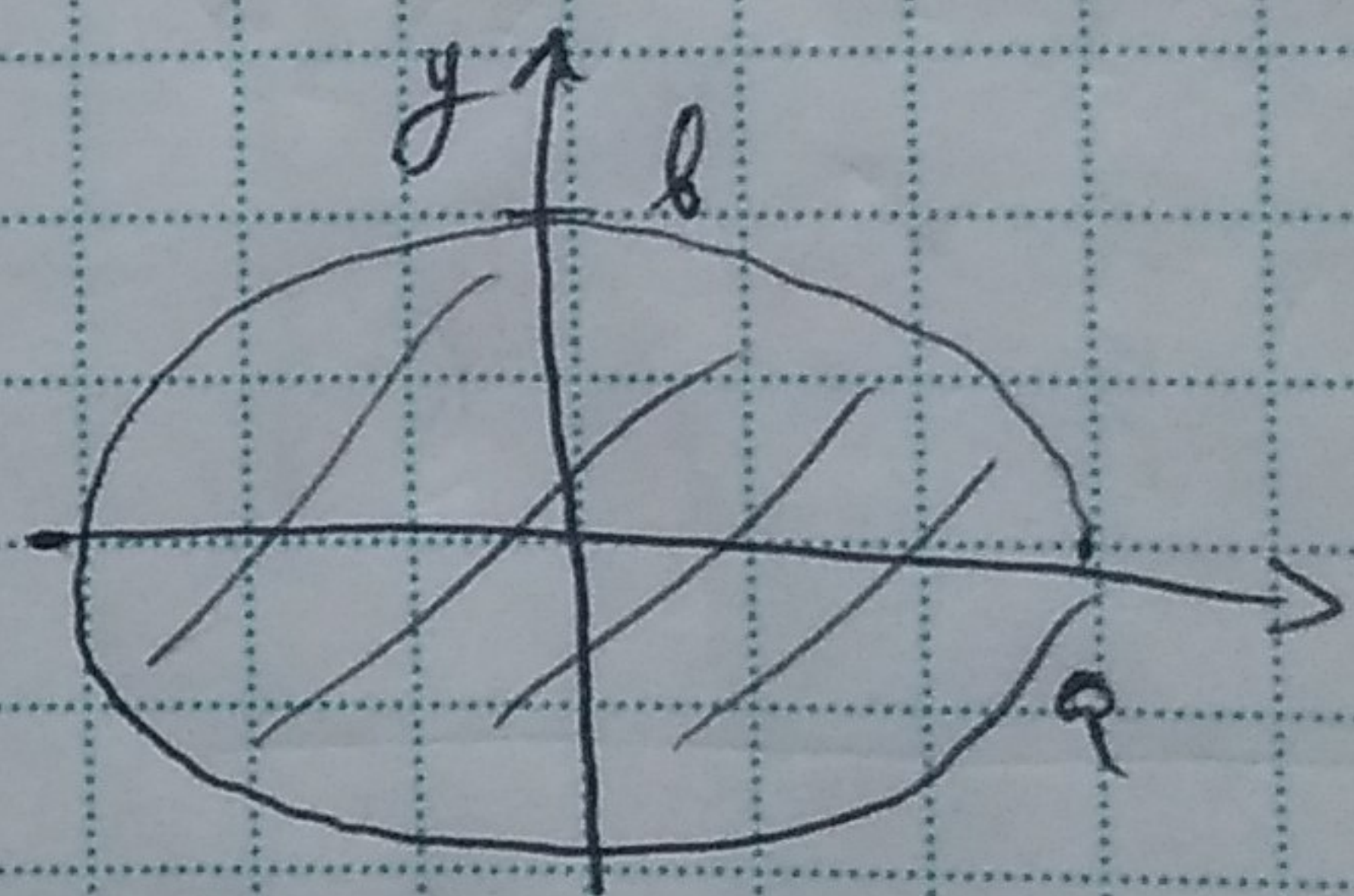
Знайти площадь

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Решение. Область симметрична относительно осей Ox и Oy .

$$y = \pm b \sqrt{1 - \frac{x^2}{a^2}}, \quad x \in [-a, a]$$

тогда $S = 4 \int_0^a b \sqrt{1 - \frac{x^2}{a^2}} dx = \frac{4|b|}{|a|} \int_0^a \sqrt{a^2 - x^2} dx =$



$$= \int_0^a x \cdot |a| \sin t \cdot |a| \cos t dt = \frac{4|b|}{|a|} \int_0^{\pi/2} |a| \cos t \cdot |a| \cos t dt = a^2 \int_0^{\pi/2} \cos^2 t dt$$

$$\frac{4|b|a^2}{|a|} \int_0^{\pi/2} (1 + \cos 2t) dt = \frac{4a^2|b|}{2|a|} \left(\frac{\pi}{2} + \left(\frac{\sin 2t}{2} \right) \Big|_0^{\pi/2} \right) = \frac{4a^2}{2} \left(\frac{\pi}{2} + 0 \right) = a^2 \frac{\pi}{2}$$

параметризуем:
 $x = a \cos t, \quad y = b \sin t, \quad t \in [0, 2\pi]$

Задача N 156 б) пусть $x = x(t), y = y(t), t \in [t_0, t_1]$, то
 $S = \int_{t_0}^{t_1} x(t) y'(t) dt = \int_{t_0}^{t_1} x'(t) y(t) dt = \frac{1}{2} \int_{t_0}^{t_1} (x(t) y'(t) - x'(t) y(t)) dt$
 тогда $S = 4 \int_0^{\pi/2} ab \cos^2 t dt = 4ab \int_0^{\pi/2} \frac{1 + \cos 2t}{2} dt = 2ab \left(\frac{\pi}{2} \right) = \pi ab$

N 160 a

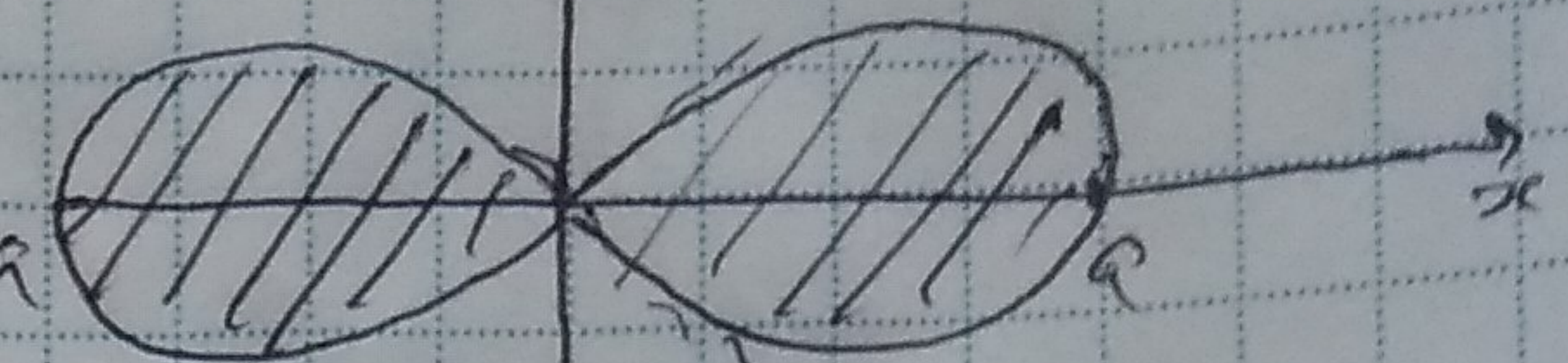
Знайти площадь фигуры, ограниченной кривыми

$$r^2 = a^2 \cos 2\varphi$$

Решение. Пусть $a > 0 \Rightarrow r = a \sqrt{\cos 2\varphi}$
 $\cos 2\varphi \geq 0 \Rightarrow -\frac{\pi}{2} \leq 2\varphi \leq \frac{\pi}{2} \Rightarrow -\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4}$

$$\frac{3\pi}{4} \leq \varphi \leq \frac{5\pi}{4}$$

$$S = \frac{1}{2} \int_{\varphi_1}^{\varphi_2} r^2 d\varphi = \int_{-\pi/4}^{\pi/4} (a \sqrt{\cos 2\varphi})^2 d\varphi =$$



⊖ / обл. симетр. / = $2a^2 \int_0^{\pi/4} (\sqrt{\cos 2\varphi})^2 d\varphi = 2a^2 \int_0^{\pi/4} |\cos 2\varphi| d\varphi =$
 $= 2a^2 \int_0^{\pi/4} \cos 2\varphi d\varphi = 2a^2 \frac{\sin 2\varphi}{2} \Big|_0^{\pi/4} = a^2$

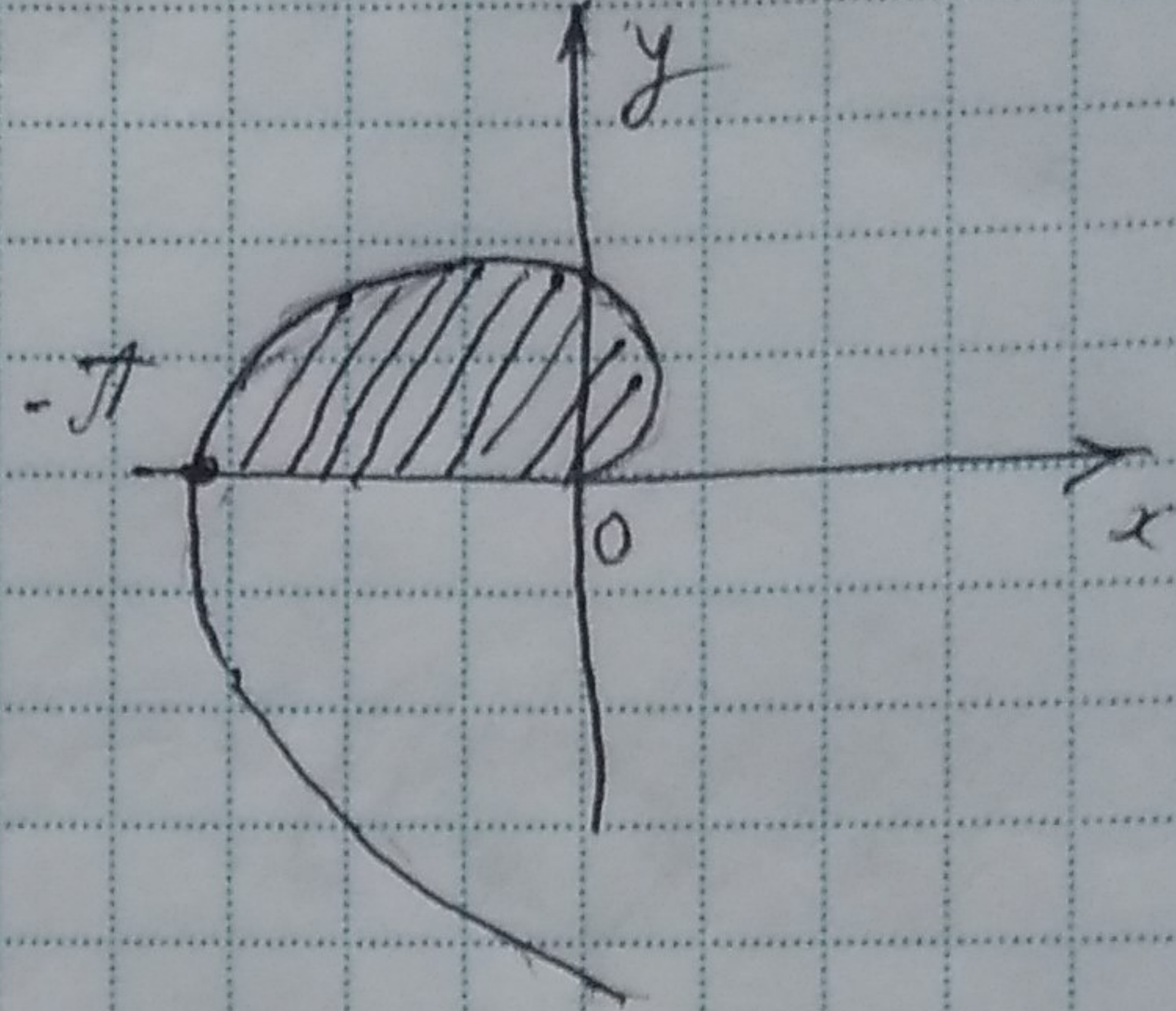
N 160 g

$$r = \varphi, \quad \varphi \in [0, \pi]$$

Решение. $\varphi \in [0, \pi]$

$$S = \frac{1}{2} \int_0^{\pi} r^2 d\varphi = \frac{1}{2} \int_0^{\pi} \varphi^2 d\varphi =$$

$$= \frac{1}{2} \cdot \frac{\varphi^3}{3} \Big|_0^{\pi} = \frac{\pi^3}{6}$$



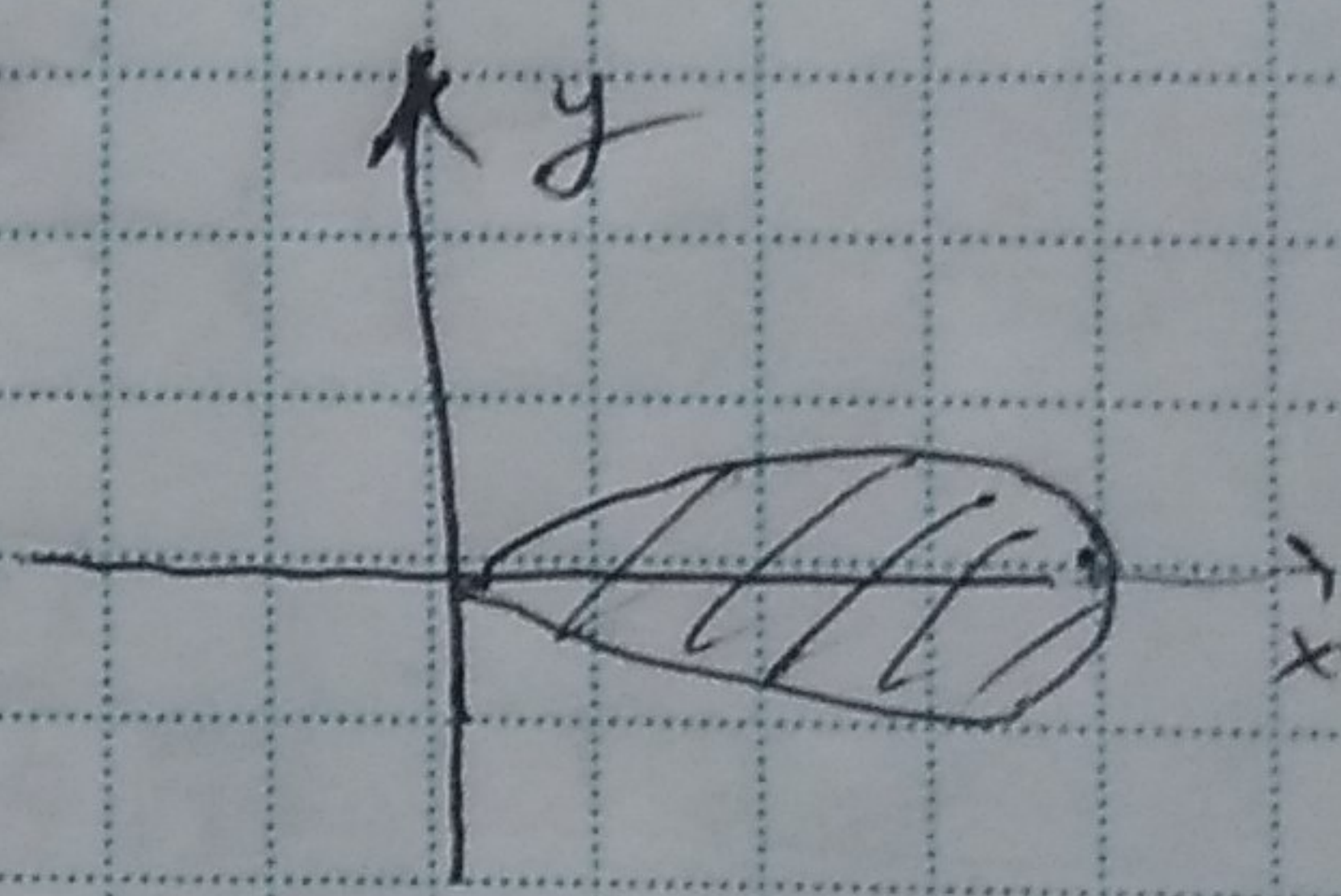
N 160 u

$$r^2 + \varphi^2 = 1$$

Решение. $r = \sqrt{1 - \varphi^2}, \quad -1 \leq \varphi \leq 1$

$$S = \frac{1}{2} \int_{-1}^1 (1 - \varphi^2) d\varphi = \int_0^1 (1 - \varphi^2) d\varphi = \left(\varphi - \frac{\varphi^3}{3} \right) \Big|_0^1 =$$

$$= \frac{2}{3}$$



N 160 u)

$$\varphi = r - \sin r, \quad \varphi \in [0, \pi]$$

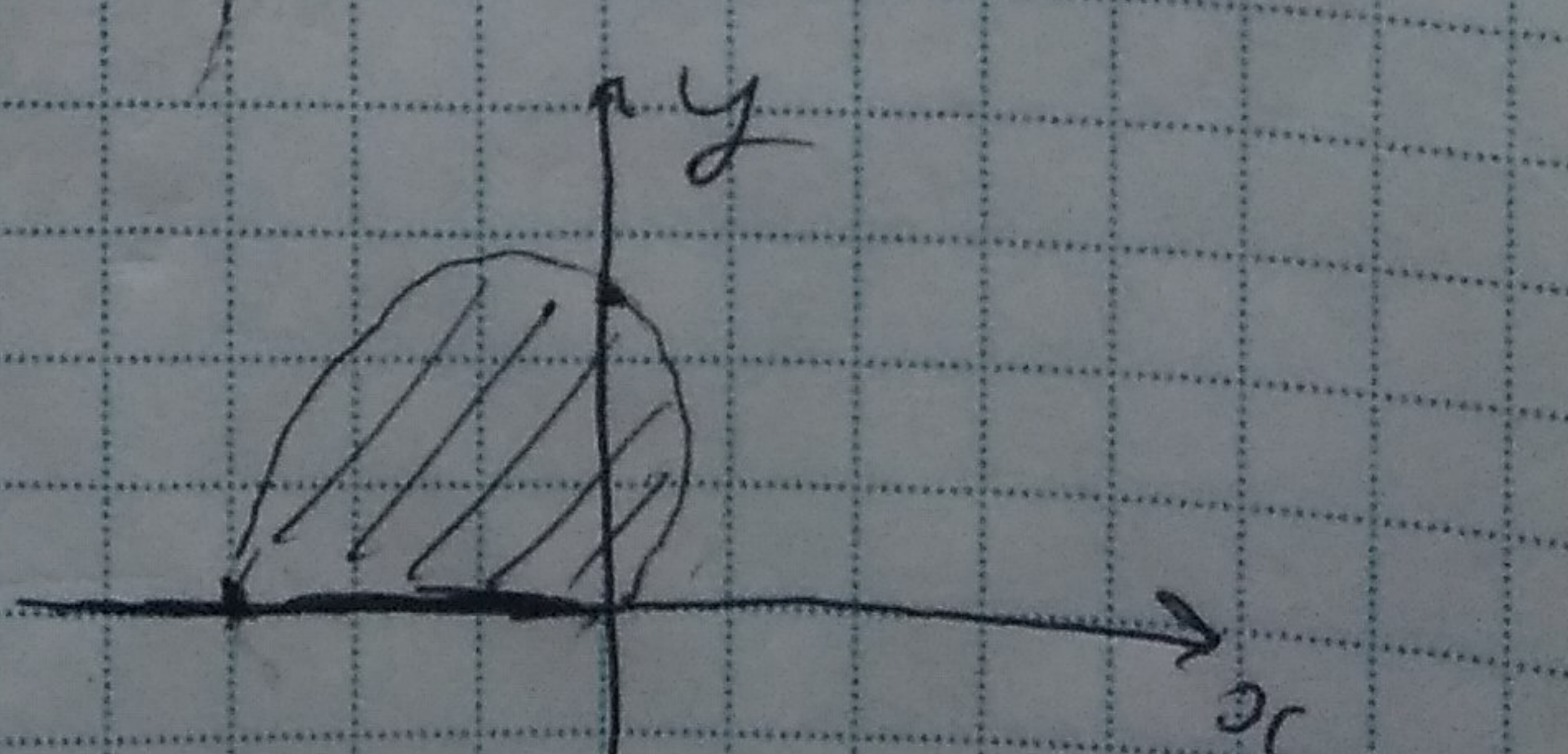
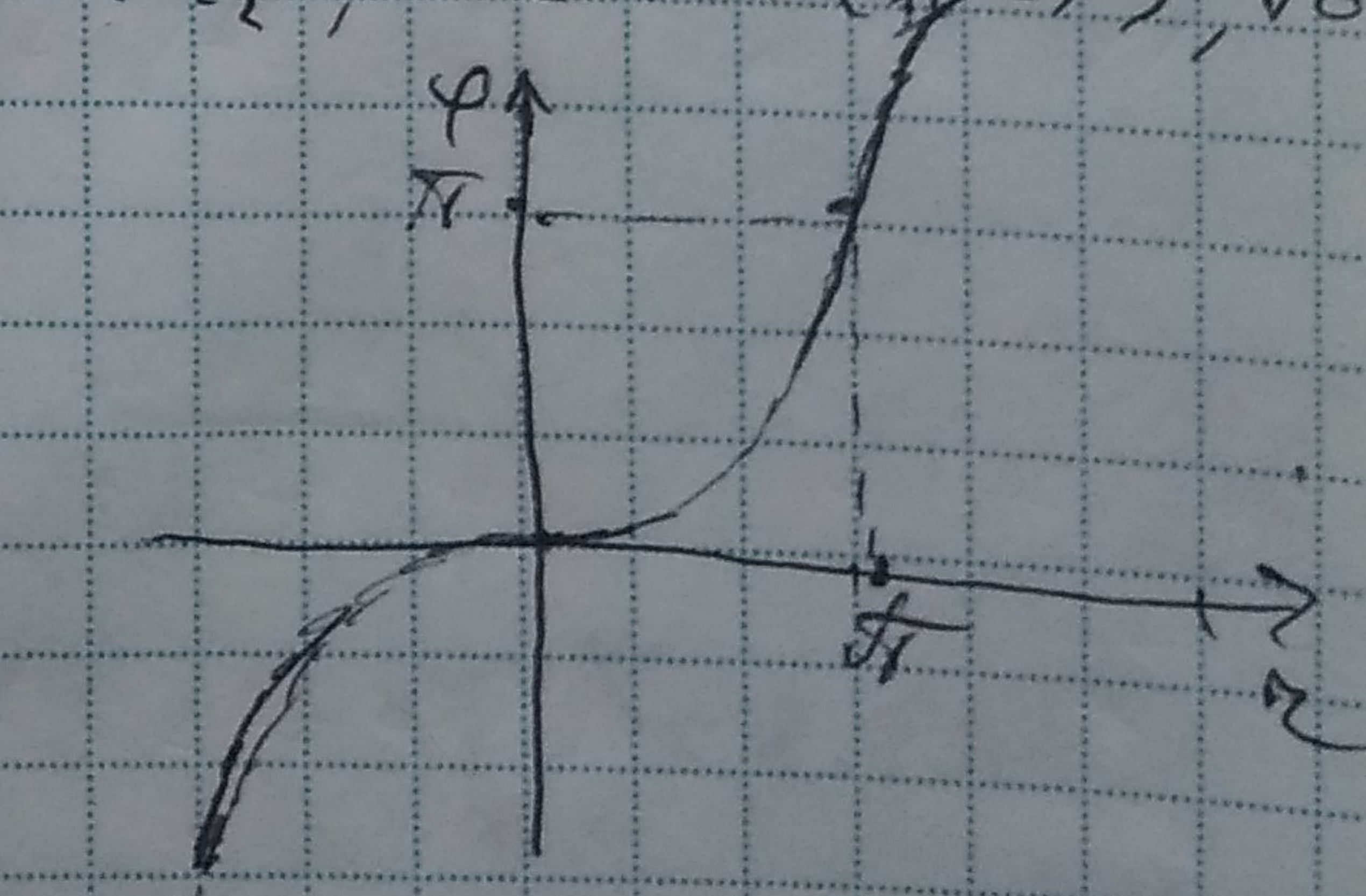
Решение. N 156 e) $\varphi = \{(r, \varphi) \mid r_1 \leq r \leq r_2, 0 \leq \varphi \leq \varphi_1/2\pi\}$, то

$$S = \frac{1}{2} \int_{r_1}^{r_2} r^2 \cdot \varphi_1'(r) dr, \quad 0 \leq r \leq \pi$$

$$\Rightarrow S = \frac{1}{2} \int_0^{\pi} r^2 (r - \sin r)' dr =$$

$$= \frac{1}{2} \int_0^{\pi} r^2 (1 - \cos r) dr = \frac{1}{2} \left(\frac{r^3}{3} \right) \Big|_0^{\pi} -$$

$$- \frac{1}{2} \int_0^{\pi} r^2 \cos r dr = \frac{\pi^3}{6} - \frac{1}{2} \gamma_1 \quad \ominus$$



$$\gamma_1 = \int_0^{\pi} u = r^2, du = \cos r dr \quad / \quad r^2 \sin r \Big|_0^{\pi} -$$

$$\int_0^{\pi} 2r \sin r dr = \int_0^{\pi} u \sin r, du = \sin r dr \quad / \quad$$

$$= -2 \left(-r \cos r \right) \Big|_0^{\pi} + \int_0^{\pi} \cos r dr = -2(\pi + 0) = -2\pi$$

$$\ominus \frac{\pi^3}{6} + \pi$$

N160 a)

$$\varphi = 4z - z^3, \quad \varphi = 0 \quad (\varphi \geq 0)$$

Решение $0 \leq z \leq 2$

$$S = \frac{1}{2} \int_{z_1}^{z_2} z^2 \varphi'(z) dz$$

$$= \frac{1}{2} \int_0^2 z^2 (4z - z^3)' dz = \frac{1}{2} \int_0^2 z^2 (4 - 3z^2) dz$$

$$= \frac{1}{2} \left(\frac{4 \cdot z^3}{3} - \frac{3 \cdot z^5}{5} \right) \Big|_0^2 = \frac{1}{2} \left(\frac{4 \cdot 8}{3} - \frac{3 \cdot 32}{5} \right) =$$

$$= \frac{1}{2} \left(\frac{32}{3} - \frac{3 \cdot 32}{5} \right) = 16 \left(\frac{1}{3} - \frac{3}{5} \right) = 16 \cdot \left(-\frac{4}{15} \right) =$$

$$= -\frac{64}{15} < 0 \quad \nearrow \text{не может быть!}$$

Видимая площадь

Для верности формулы Факон
поделим, чтоб $\varphi \uparrow$ (зрост)

$$\text{То же } S = \frac{1}{2} \left(\int_0^{2/\sqrt{3}} z^2 (4z - z^3)' dz - \int_0^2 z^2 (4z - z^3)' dz \right) = \frac{1}{2} \left(\int_0^{2/\sqrt{3}} (4z^2 - 3z^4) dz - \int_0^2 (4z^2 - 3z^4) dz \right)$$

$$= \frac{1}{2} \left(\left(\frac{4z^3}{3} - \frac{3z^5}{5} \right) \Big|_0^{2/\sqrt{3}} - \left(\frac{4z^3}{3} - \frac{3z^5}{5} \right) \Big|_0^2 \right) =$$

$$= \frac{1}{2} \left(\left(\frac{32}{9\sqrt{3}} - \frac{3 \cdot 32}{5 \cdot 9\sqrt{3}} \right) - \left(\frac{32}{3} - \frac{3 \cdot 32}{5} \right) \right) =$$

$$= \frac{32}{9\sqrt{3}} \left(1 - \frac{3}{5} \right) - \frac{1}{2} \left(\frac{32}{3} - \frac{3 \cdot 32}{5} \right) = \frac{32 \cdot 2}{9 \cdot 5 \cdot \sqrt{3}} + \frac{32}{15} = \frac{32}{15} \left(\frac{2}{3\sqrt{3}} + 1 \right) = \frac{64}{15} \left(\frac{1}{3\sqrt{3}} + 1 \right)$$

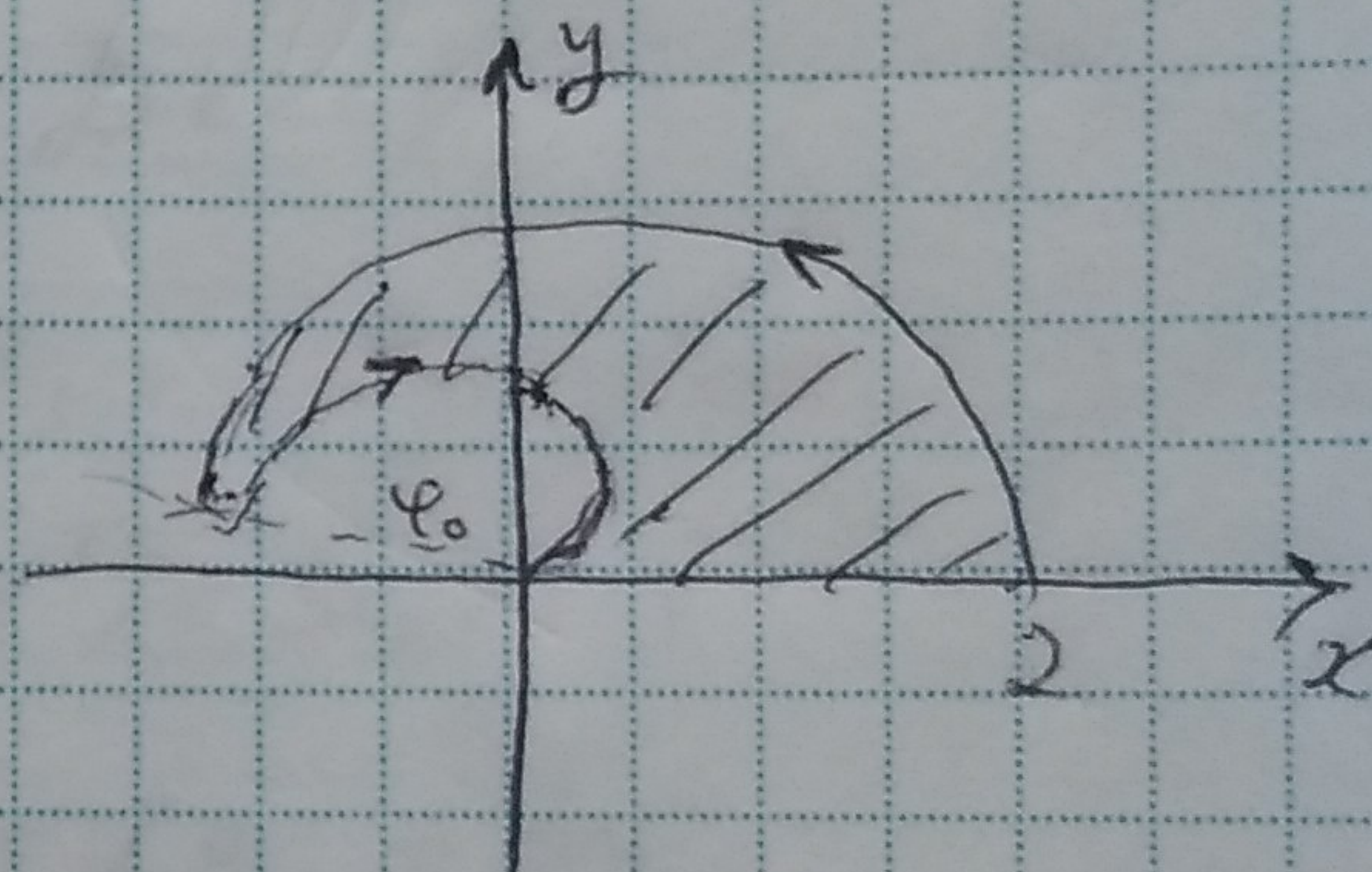
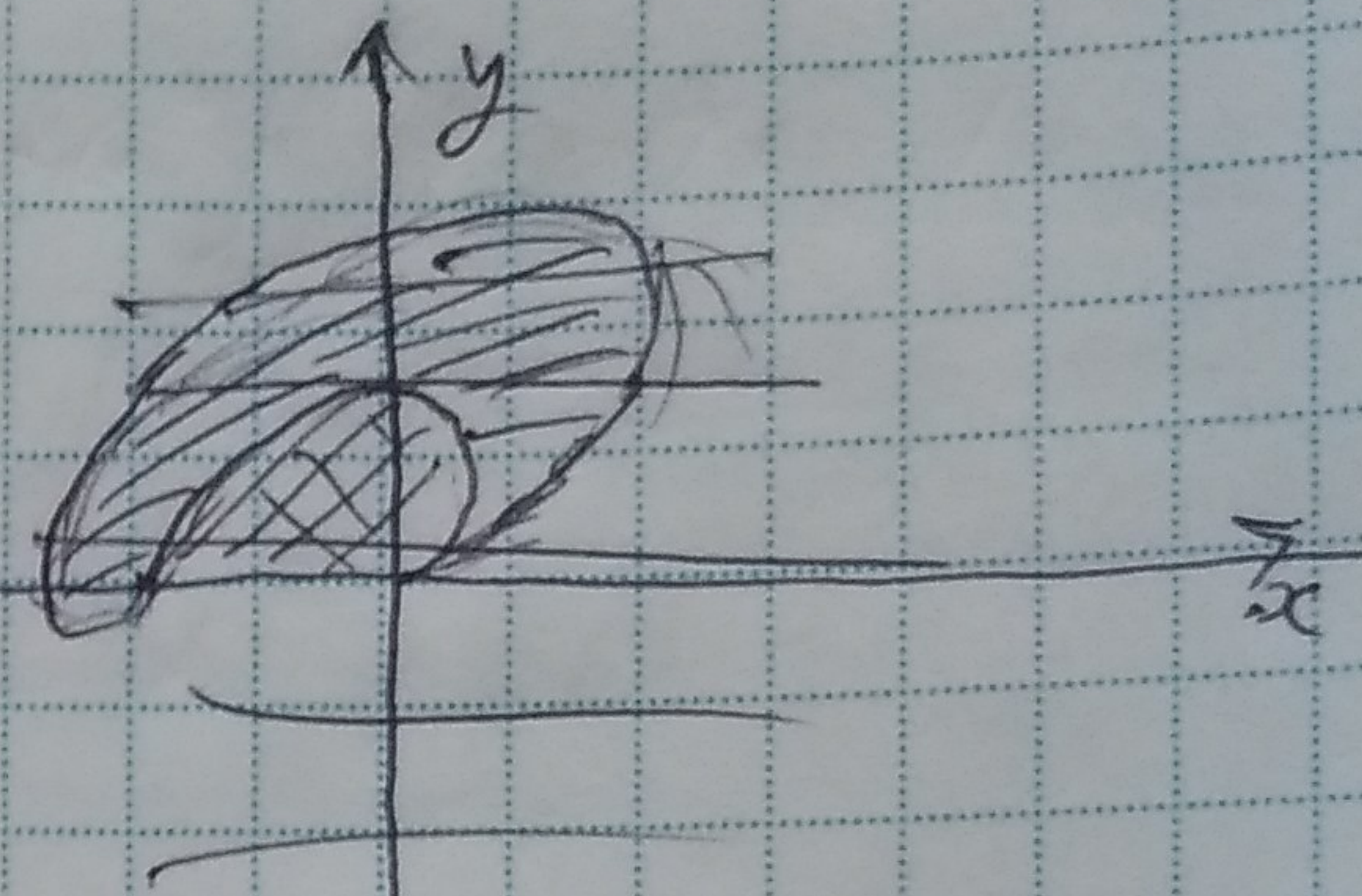
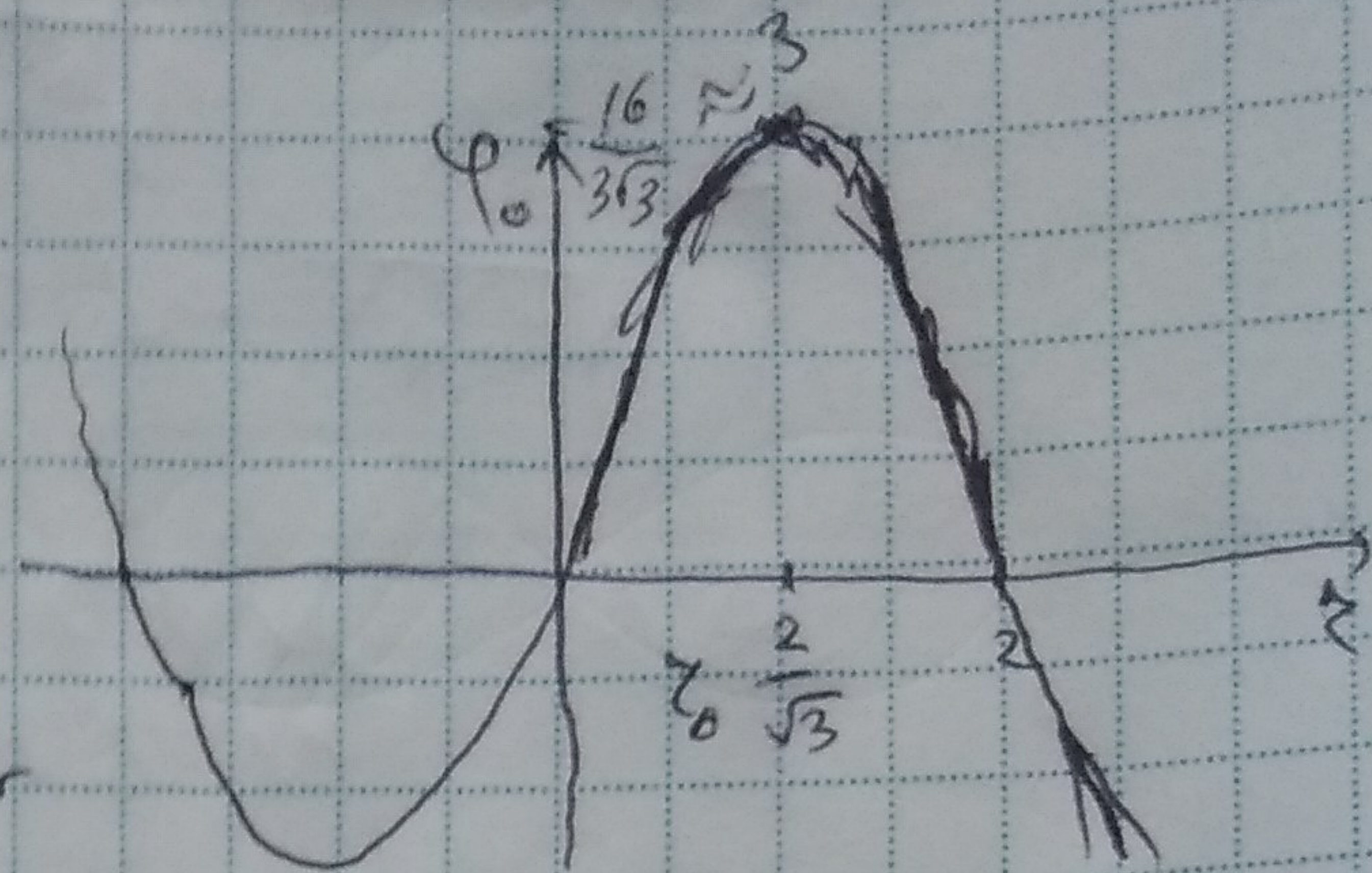
N160 б)

$$x = a(t - \sin t), \quad y = a(1 - \cos t), \quad y \geq 0, \quad (0 \leq t \leq 2\pi)$$

Решение. Изучим задачу N156 б)

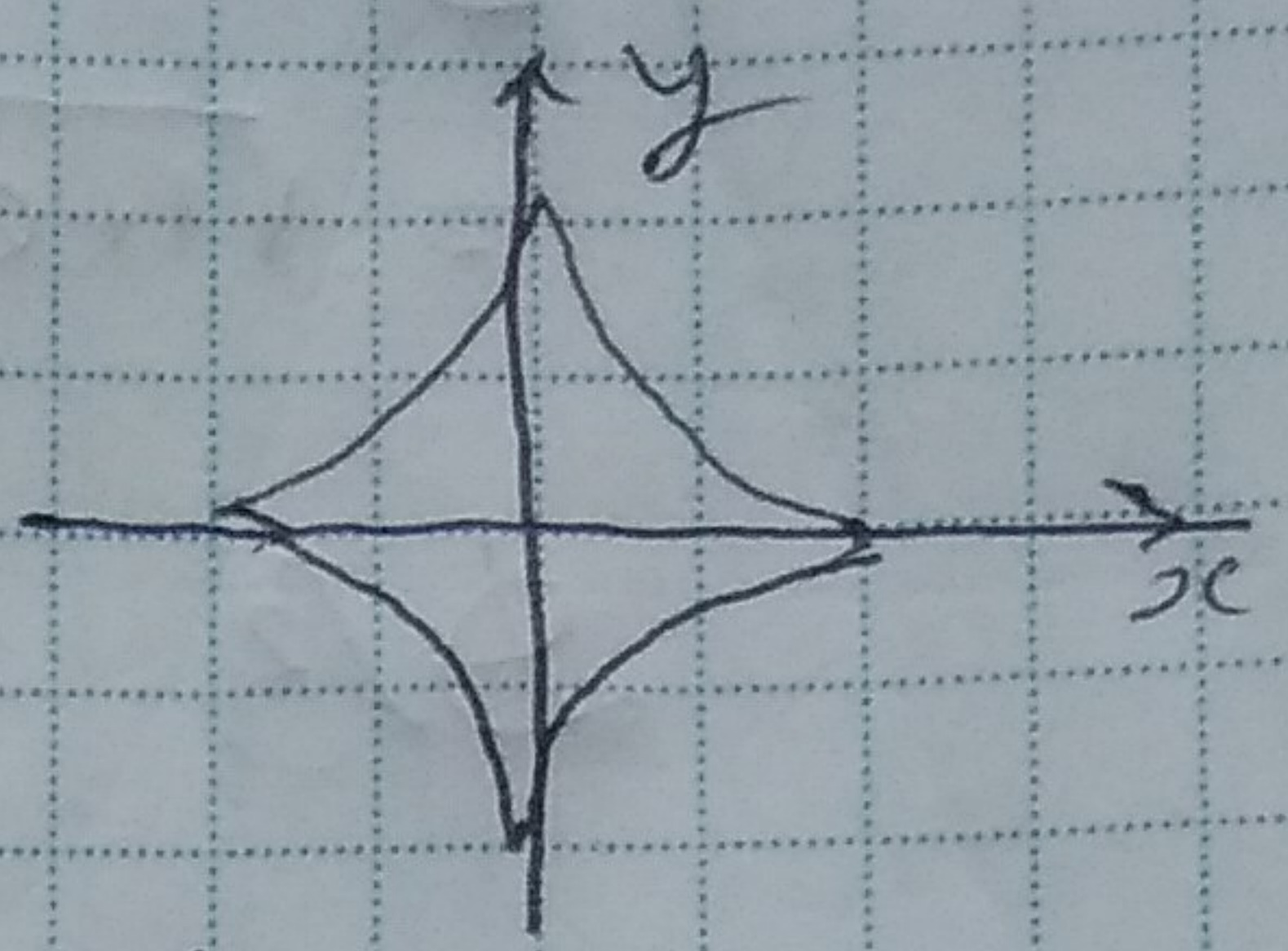
$$S = \left| \int_0^{2\pi} x'(t) \cdot y(t) dt \right| = \int_0^{2\pi} a^2 (1 - \cos t)^2 dt =$$

$$= a^2 \int_0^{2\pi} \left(1 - 2\cos t + \frac{1 + \cos 2t}{2} \right) dt = a^2 \left(t + \frac{1}{2} t \right) \Big|_0^{2\pi} = 3\pi a^2$$



N 162 m) Знаючи наочну, однею нелінійною кривою $x = \cos^3 t$, $y = \sin^3 t$

Розв'язок. $t \in [0, 2\pi]$



$$\begin{aligned}
 S &= \frac{1}{2} \int_0^{2\pi} (x(t)y'(t) - x'(t)y(t)) dt = \\
 &= \text{площа кривої} / 2 = 2 \int_0^{\pi/2} (\cos^3 t \cdot (\sin^3 t)' - (\cos^3 t)' \cdot \sin^3 t) dt = \\
 &= 6 \int_0^{\pi/2} (\cos^3 t \cdot \sin^2 t \cdot \cos t + \cos^2 t \cdot \sin^4 t \cdot \sin t) dt = \\
 &= 6 \int_0^{\pi/2} (\cos^4 t \cdot \sin^2 t + \cos^2 t \sin^5 t) dt = 6 \int_0^{\pi/2} \cos^2 t \sin^2 t dt = \\
 &= \frac{3}{2} \int_0^{\pi/2} \sin^2 2t dt = \frac{3}{2} \int_0^{\pi/2} \frac{1 - \cos 4t}{2} dt = \frac{3}{4} \left(t - \frac{\sin 4t}{4} \right) \Big|_0^{\pi/2} \\
 &= \frac{3}{4} \left(\frac{\pi}{2} \right) = \frac{3\pi}{8}
 \end{aligned}$$

N 162 y) $x^3 + y^3 = 3axy$. (Крива Дікарта)

Розв'язок. Параметризуємо криву. Заміна: $y = tx$, тоді $x^3 + t^3 x^3 = 3axtx \Rightarrow x(1+t^3) = 3at \Rightarrow$

$$\Rightarrow x = \frac{3at}{1+t^3}, \quad y = \frac{3at^2}{1+t^3}, \quad t \in [0, +\infty)$$

$$\begin{aligned}
 S &= \frac{1}{2} \int_0^{+\infty} \left(\frac{3at}{1+t^3} \cdot \frac{6at(1+t^3) - 3t^2 \cdot 3at^2}{(1+t^3)^2} - \frac{3at^2}{1+t^3} \cdot \frac{3a(1+t^3) - 3at \cdot 3t^2}{(1+t^3)^2} \right) dt \\
 &= \frac{1}{2} \int_0^{+\infty} \frac{3at \cdot (6at - 3at^4) - 3at^2(3a - 6at^3)}{(1+t^3)^3} dt = \\
 &= \frac{9a^2}{2} \int_0^{+\infty} \frac{t^2(2-t^3) - t^2(1-2t^3)}{(1+t^3)^3} dt = \frac{9a^2}{2} \int_0^{+\infty} \frac{2t - t^4 - t^2 + 2t^5}{(1+t^3)^3} dt \\
 &= \frac{9a^2}{2} \int_0^{+\infty} \frac{t^2(1+t^3)}{(1+t^3)^3} dt = \frac{9a^2}{2} \int_0^{+\infty} \frac{t^2}{(1+t^3)^2} dt \\
 &= \frac{3a^2}{2} \int_0^{+\infty} \frac{d(1+t^3)}{(1+t^3)^2} = -\frac{3a^2}{2} \frac{1}{1+t^3} \Big|_0^{+\infty} = \frac{3a^2}{2}
 \end{aligned}$$

N 163 b) Обчислити довжину дуги $y = e^x$, $0 \leq x \leq 1$

$$\begin{aligned}
 \text{Розв'язок. } L &= \int_{x_1}^{x_2} \sqrt{1 + (y'(x))^2} dx = \int_0^1 \sqrt{1 + e^{2x}} dx = \\
 &= \int_{x=\ln t}^{x=1} \sqrt{1+t^2} \frac{dx=dt}{t} = \int_1^e \frac{\sqrt{1+t^2}}{t} dt = \int_1^e t^{-1} (1+t^2)^{1/2} dt = \\
 &= \int_1^e t^{-2} (t^2+1)^{1/2} dt = \int_1^e \frac{1}{t} \sqrt{1+t^2} dt \quad \begin{matrix} 2\text{-а заміна } yed. \\ 1+t^2 = y^2 \end{matrix} \\
 &= \int_{t=\sqrt{y^2-1}}^{t=y} \frac{y}{\sqrt{y^2-1}} dy
 \end{aligned}$$

$$= \int_2^{\sqrt{1+e^2}} \frac{y^2}{y^2-1} dy = \int_2^{\sqrt{1+e^2}} \left(1 + \frac{1}{y^2-1}\right) dy = \left(y - \frac{1}{2} \ln \left| \frac{y-1}{y+1} \right| \right) \Big|_2^{\sqrt{1+e^2}}$$

$$= \sqrt{1+e^2} - 2 - \frac{1}{2} \ln \left| \frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1} \right| + \frac{1}{2} \ln \left| \frac{1}{3} \right|$$

Д/З N 159 б) H1, N 160 г) x) K) H), N 161 б)
 N 162 а) H), N 163 г), ~~N 165 а)~~